Constraints on Higher-Order Perturbative Corrections in $b \to u$ Semileptonic Decays from Residual Renormalization-Scale Dependence

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Abstract

The constraint of a progressive decrease in residual renormalization scale dependence with increasing loop order is developed as a method for obtaining bounds on unknown higher-order perturbative corrections to renormalization-group invariant quantities. This technique is applied to the inclusive semileptonic process $b \to u\bar{\nu}_\ell \ell^-$ (explicitly known to two-loop order) to obtain bounds on the three- and four-loop perturbative coefficients that are not accessible via the renormalization group. Using the principle of minimal sensitivity, an estimate is obtained for the perturbative contributions to $\Gamma\left(b \to u\bar{\nu}_\ell \ell^-\right)$ that incorporates theoretical uncertainty from as-yet-undetermined higher order QCD corrections.

A variety of techniques have been developed for obtaining estimates of higher-loop corrections in perturbative QCD, including the principle of minimal sensitivity (PMS) [1], the fastest apparent convergence (or effective charges) approach [2], nonabelianization methods [3], Padé approximations [4], and renormalization-group supplemented Padé approximants [5]. Of particular relevance is the PMS ansatz which proposes that renormalization-scheme independence, which at lower-loop levels translates into renormalization-scale (μ) independence, provides the optimum perturbative prediction of a QCD observable.

In a particular renormalization scheme (such as $\overline{\text{MS}}$), the minimal sensitivity principle identifies the appropriate choice of renormalization scale μ for a physical observable as the value at which the observable is independent of μ, providing a method for dealing with the residual renormalization scale dependence that exists in a perturbative calculation truncated to any given order. In explicit calculations, such residual scale dependence decreases as higher-order corrections are included, as one would expect perturbative predictions to become virtually μ-independent at sufficiently high orders. For example, in inclusive semileptonic $b \to u$ decays, a clear progressive flattening of the decay rate as a function of μ is observed in the explicit calculations up to two-loop order [6], and this property persists when a Padé estimate of the three-loop correction is included [7].

In this paper we demonstrate for semi-leptonic $b \to u$ decays that the progressive decrease in renormalization-scale dependence with increasing loop-order places meaningful bounds on unknown higher-order perturbative coefficients. Furthermore, the minimal-sensitivity prediction of the decay rate devolving from these bounds is obtained, illustrating the phenomenological utility of this approach in estimating higher-loop QCD effects on standard-model observables.

The perturbative contributions to the inclusive semileptonic $b \to u$ decay rate $\Gamma(b \to u\bar{\nu}_\ell \ell^-)$ may be expressed

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as [6]

$$\frac{1}{K}\Gamma\left(\mu, m_b(\mu), x(\mu)\right) = m_b^5(\mu)\Big(1+\left[a_0 - a_1\log(w)\right]x(\mu) + \left[b_0 - b_1\log(w) + b_2\log^2(w)\right]x^2(\mu)$$
$$+\left[c_0 - c_1\log(w) + c_2\log^2(w) - c_3\log^3(w)\right]x^3(\mu) \tag{1}$$
$$+\left[d_0 - d_1\log(w) + d_2\log^2(w) - d_3\log^3(w) + d_4\log^4(w)\right]x^4(\mu) + \ldots\Big)$$

where

$$x(\mu) \equiv \frac{\alpha_s(\mu)}{\pi} \quad , \quad w = w(\mu, m_b(\mu)) \equiv \frac{m_b^2(\mu)}{\mu^2} \quad , \quad K \equiv \frac{G_F^2\left|V_{ub}\right|^2}{192\pi^3} \quad . \tag{2}$$

In the $\overline{\mathrm{MS}}$ scheme in which m_b is identified with a scale-dependent (running) b-quark mass for four or five active flavours, the one- and two-loop order coefficients within (1) are given by [6]

$$\begin{aligned}
&\text{all } n_f: \quad a_0 = 4.25360 \quad , \quad a_1 = 5 \quad , \\
&n_f = 5: \quad b_0 = 26.7848 \quad , \quad b_1 = 36.9902 \quad , \quad b_2 = 17.2917 \quad , \\
&n_f = 4: \quad b_0 = 25.7547 \quad , \quad b_1 = 38.3935 \quad , \quad b_2 = 17.7083 \quad .
\end{aligned} \tag{3}$$

The running mass $m_b(\mu)$ appearing in (1) is also known to full four-loop order [9]:

$$m_b\left[x(\mu)\right] = m_b\left[x\left(\mu_0\right)\right]\frac{c\left[x(\mu)\right]}{c\left[x\left(\mu_0\right)\right]} \quad , \tag{4}$$

$$n_f = 4: \quad c[x] = x^{12/25}\left[1 + 1.01413x + 1.38920x^2 + 1.09052x^3 + \ldots\right], \tag{5}$$

$$n_f = 5: \quad c[x] = x^{12/23}\left[1 + 1.17549x + 1.50071x^2 + 0.172486x^3 + \ldots\right] \quad . \tag{6}$$

Renormalization-group invariance of the decay rate Γ determines a subset of the unknown three-loop c_k coefficients and four-loop d_k coefficients. From the renormalization-group equation

$$\begin{aligned}
0 &= \mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}\Gamma\left(\mu, m_b(\mu), x(\mu)\right) \\
&= \left[\mu^2\frac{\partial}{\partial\mu^2} - \left(\gamma_0 x + \gamma_1 x^2 + \gamma_2 x^3 + \ldots\right)m_b\frac{\partial}{\partial m_b} - \left(\beta_0 x^2 + \beta_1 x^3 + \ldots\right)\frac{\partial}{\partial x}\right]\Gamma \quad ,
\end{aligned} \tag{7}$$

we find the following expressions for terms proportional to x^3 and x^4 in the final line of (7):

$$c_1 = 2b_0\beta_0 + a_0\beta_1 + \gamma_0\left(5b_0 - 2b_1\right) + \gamma_1\left(5a_0 - 2a_1\right) + 5\gamma_2 \quad , \tag{8}$$

$$c_2 = \frac{1}{2}\left[2b_1\beta_0 + a_1\left(\beta_1 + 5\gamma_1\right) + \gamma_0\left(5b_1 - 4b_2\right)\right] \quad , \tag{9}$$

$$c_3 = \frac{b_2}{3}\left(2\beta_0 + 5\gamma_0\right) \quad , \tag{10}$$

$$d_1 = \beta_2 a_0 + 2\beta_1 b_0 + 3\beta_0 c_0 + \gamma_0\left(5c_0 - 2c_1\right) + \gamma_1\left(5b_0 - 2b_1\right) + \gamma_2\left(5a_0 - 2a_1\right) + 5\gamma_3 \tag{11}$$

$$d_2 = \frac{1}{2}\beta_2 a_1 + \beta_1 b_1 + \frac{3}{2}\beta_0 c_1 + \gamma_0\left(\frac{5}{2}c_1 - 2c_2\right) + \gamma_1\left(\frac{5}{2}b_1 - 2b_2\right) + \frac{5}{2}\gamma_2 a_1 \tag{12}$$

$$d_3 = \beta_0 c_2 + \frac{2}{3}\beta_1 b_2 + \gamma_0\left(\frac{5}{3}c_2 - 2c_3\right) + \frac{5}{3}\gamma_1 b_2 \tag{13}$$

$$d_4 = \frac{3}{4}\beta_0 c_3 + \frac{5}{4}\gamma_0 c_3 \tag{14}$$

Upon substitution of (3) and the four-loop $\overline{\text{MS}}$ results for the β function [8] and anomalous mass dimension γ [9] into (8–14), we obtain the following numerical values for these higher-loop coefficients:

$$c_1^{(4)} = 263.839\ ,\ c_2^{(4)} = 194.234\ ,\ c_3^{(4)} = 54.1087 \tag{15}$$

$$d_1^{(4)} = 11.25c_0^{(4)} + 103.081\ ,\ d_2^{(4)} = 1580.26\ ,\ d_3^{(4)} = 765.844\ ,\ d_4^{(4)} = 152.181 \tag{16}$$

$$c_1^{(5)} = 249.592\ ,\quad c_2^{(5)} = 178.755\ ,\quad c_3^{(5)} = 50.9145 \tag{17}$$

$$d_1^{(5)} = 10.75c_0^{(5)} - 8.28683\ ,\ d_2^{(5)} = 1376.68\ ,\ d_3^{(5)} = 667.838\ ,\ d_4^{(5)} = 136.833 \tag{18}$$

where the superscript denotes the number of active flavours (either $n_f = 4$ or $n_f = 5$). In the energy region spanning the threshold between $n_f = 4$ and $n_f = 5$ there are four higher-loop parameters that remain undetermined: $\left\{c_0^{(4)},\ c_0^{(5)},\ d_0^{(4)},\ d_0^{(5)}\right\}$. However, continuity of the decay rate at this threshold in conjunction with the coupling-constant and quark-mass threshold matching conditions imposed at $\mu = m_b$ [10],[1]

$$x^{(4)}(m_b) = x^{(5)}(m_b)\left[1 + 0.1528\left[x^{(5)}(m_b)\right]^2 + 0.633\left[x^{(5)}(m_b)\right]^3\right] \tag{19}$$

$$m_b^{(4)}(m_b) = m_b^{(5)}(m_b)\left[1 + 0.2060\left[x^{(5)}(m_b)\right]^2 + 1.9464\left[x^{(5)}(m_b)\right]^3\right]\quad , \tag{20}$$

effectively reduces this set to two unknown parameters. Using the benchmark value $\alpha_s(M_z) = 0.119$ [11], $m_b = 4.2\,\text{GeV}$ [12], and the four-loop β function, we first find that

$$x^{(5)}(4.2\,\text{GeV}) = 0.07261 \quad , \quad x^{(4)}(4.2\,\text{GeV}) = 0.07269 \tag{21}$$

$$m_b^{(4)}(4.2\,\text{GeV}) = 4.208\,\text{GeV} \quad . \tag{22}$$

Imposing continuity of the decay rates to $\mathcal{O}\left(x^3\right)$ at this flavour threshold then yields

$$c_0^{(5)} = 1.012c_0^{(4)} + 15.66 \quad . \tag{23}$$

Such a continuity condition cannot be extended to $\mathcal{O}\left(x^4\right)$, because the first unknown term in the mass threshold condition (20) contributes at $\mathcal{O}\left(x^4\right)$ to the d_0 coefficient. In the absence of this information, we assume

$$d_0^{(4)} \approx d_0^{(5)} \quad , \tag{24}$$

reflecting the near-equivalence of the known coefficients $\{d_2,\ d_3,\ d_4\}$ for four and five active flavours.[2] Thus the RG analysis combined with continuity of the decay rate at flavour thresholds implies that the four-loop expression (1) effectively has only two undetermined parameters $\left\{c_0^{(4)},\ d_0^{(4)}\right\}$ in the energy range spanning the threshold between $n_f = 4$ and $n_f = 5$.

The decay rate Γ is a truncated perturbation series which necessarily exhibits residual renormalization scale dependence. In general if Γ is known to $\mathcal{O}(x^n)$, then $\mathrm{d}\Gamma/\mathrm{d}\mu = \mathcal{O}\left(x^{n+1}\right)$. Consequently, the residual scale dependence diminishes with increasing loop order. A measure of the residual scale dependence in the natural energy region $m_b/2 < \mu < 2m_b$ is provided by the deviation of Γ from a constant value

$$\chi^2 = \frac{1}{\frac{3}{2}m_b}\int\limits_{m_b/2}^{2m_b}\left(\frac{\Gamma(\mu)}{\langle\Gamma\rangle} - 1\right)^2 \mathrm{d}\mu \quad , \tag{25}$$

[1] In the $\overline{MS}$ scheme the scale $\mu = m_b$ is defined by $m_b(m_b) = m_b$.

[2] In (20), if the coefficient of $\left[x^{(5)}(m_b)\right]^4$ were 20, reflecting a factor of ten increase between the two previous orders, the difference between $d_0^{(4)}$ and $d_0^{(5)}$ would be approximately 150, which is small compared to the scales of d_0 we obtain in our analysis below.

where $\langle\Gamma\rangle$ is the average value of Γ over this energy interval

$$\langle\Gamma\rangle = \frac{1}{\frac{3}{2}m_b}\int\limits_{m_b/2}^{2m_b}\Gamma(\mu)\,\mathrm{d}\mu \quad , \tag{26}$$

and the pre-factor of $3m_b/2$ leads to a dimensionless χ^2. The progressive decrease in Γ's scale dependence implies that χ^2 must decrease as the loop order of Γ is increased. However, since at three-loop order Γ (and hence χ^2) will depend on the parameter $c_0^{(4)}$, and at four-loop order will depend on $\left\{c_0^{(4)},\ d_0^{(4)}\right\}$, the progressive decrease in χ^2 as loop-order increases necessarily provides constraints on these unknown higher-loop parameters.

To obtain such constraints, we use the central values $\alpha_s(M_z) = 0.119$ [11], $m_b = 4.2\,\mathrm{GeV}$ [12], the four-loop β and γ functions, and the threshold matching conditions (19), (20) to evaluate $\alpha(\mu)$ and $m_b(\mu)$. We then utilize discretization to evaluate the integrals (25) and (26) and obtain χ^2 values shown in Figure 1. The two-loop result is by definition independent of $c_0^{(4)}$; the three-loop result depends on $c_0^{(4)}$ as displayed in Figure 1. The requirement that the three-loop χ^2 is less than the two-loop χ^2 is satisfied provided

$$-150 < c_0^{(4)} < 290 \quad . \tag{27}$$

The Padé estimate $c_0^{(4)} = 188$ of [7] is well within this interval. Combining the result (27) with (23) we find the following constraint on $c_0^{(5)}$:

$$-136 < c_0^{(5)} < 309 \quad , \tag{28}$$

which, in the notation $c_0^{(5)} = 200 + \Delta$, corresponds to a range $-336 < \Delta < 109$ that includes all estimates of Δ discussed in [13]. The optimal value of $c_0^{(4)}$, *i.e.*, the value that minimizes χ^2, occurs at $c_0^{(4)} = 57$ ($c_0^{(5)} = 74$) corresponding most closely to the large-β_0 estimate ($\Delta = -100$) of [13, 14].

The progressive decrease in χ^2 to four-loop order restricts $\left\{c_0^{(4)},\ d_0^{(4)}\right\}$ parameter space to the region indicated in Figure 2. Further insight into the allowed $d_0^{(4)}$ range can be obtained by employing the Padé estimate $c_0^{(4)} = 188$ [7] to determine a constant three-loop value for χ^2. Upon comparison with the four-loop χ^2 dependence on the unknown parameter $d_0^{(4)}$, we find from Figure 3 that χ^2 continues to decrease with increasing order provided

$$-500 \lesssim d_0^{(4)} \lesssim 2500 \quad . \tag{29}$$

The range of parameter space covered by Figure 2 provides control over higher-order corrections in the prediction of the actual phenomenological decay rate. If we extract the minimal-sensitivity prediction of the decay rate over the entire parameter space of Figure 2, [3] we obtain the following range for the perturbative contributions to the inclusive semileptonic decay rate:

$$\frac{\Gamma}{K} = 2050 \pm 270\,\mathrm{GeV}^5 \quad . \tag{30}$$

By contrast, the perturbative two-loop rate, obtained by truncation of the series (1) after its (fully known) $\mathcal{O}\left(x^2\right)$ contributions, does not exhibit minimal sensitivity at all in the $m_b/2 \le \mu \le 2m_b$ region, but is seen to decrease with increasing μ from $2053\,\mathrm{GeV}^5$ to $1622\,\mathrm{GeV}^5$. Indeed, one unanticipated effect of including three- and four-loop coefficients, both known $\{c_{1\text{–}3},\ d_{2\text{–}4}\}$ and unknown, is to ensure that minimal sensitivity occurs in the range $m_b/2 \le \mu \le 2m_b$ over almost all of the parameter space of Figure 2.

The central value of the bound (30) corroborates the Padé estimate $2071\,\mathrm{GeV}^5$ obtained in [7]. The $\pm13\%$ uncertainty in the rate (30) is a genuine reflection of the uncertainty following from unknown higher-order QCD corrections, which typically are truncated away in phenomenological perturbative estimates of empirical quantities.

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[3]Minimal sensitivity occurs when $\mathrm{d}\Gamma/\mathrm{d}\mu = 0$ in the region $m_b/2 \le \mu \le 2m_b$. When there exists more than one critical point, we choose the one with the smallest second derivative, corresponding to the least sensitive choice.

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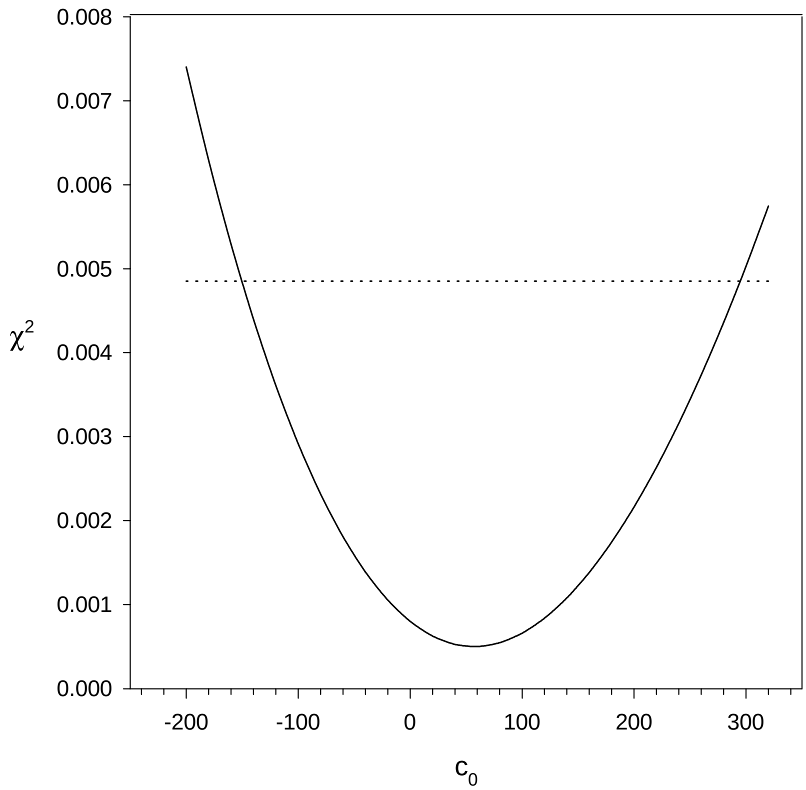

Figure 1: The quantity χ^2 as a function of $c_0^{(4)}$ (solid curve) for the three-loop contributions to Γ. The straight dashed line represents the two-loop χ^2, and the criterion that the three-loop χ^2 is smaller than the two-loop χ^2 constrains $c_0^{(4)}$ to the region between the intersection points of the three- and two-loop curves.

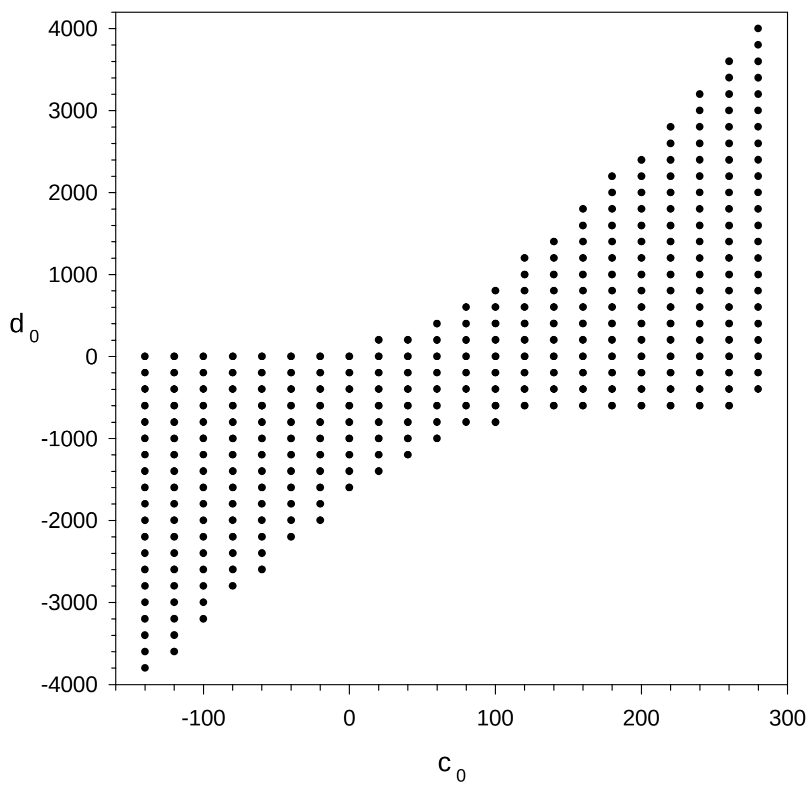

Figure 2: The dots correspond to points within $\left\{c_0^{(4)},\ d_0^{(4)}\right\}$ parameter space for which χ^2 decreases as loop-order increases from two to four. The sharp cutoff in the $c_0^{(4)}$ direction corresponds to the range obtained from Figure 1.

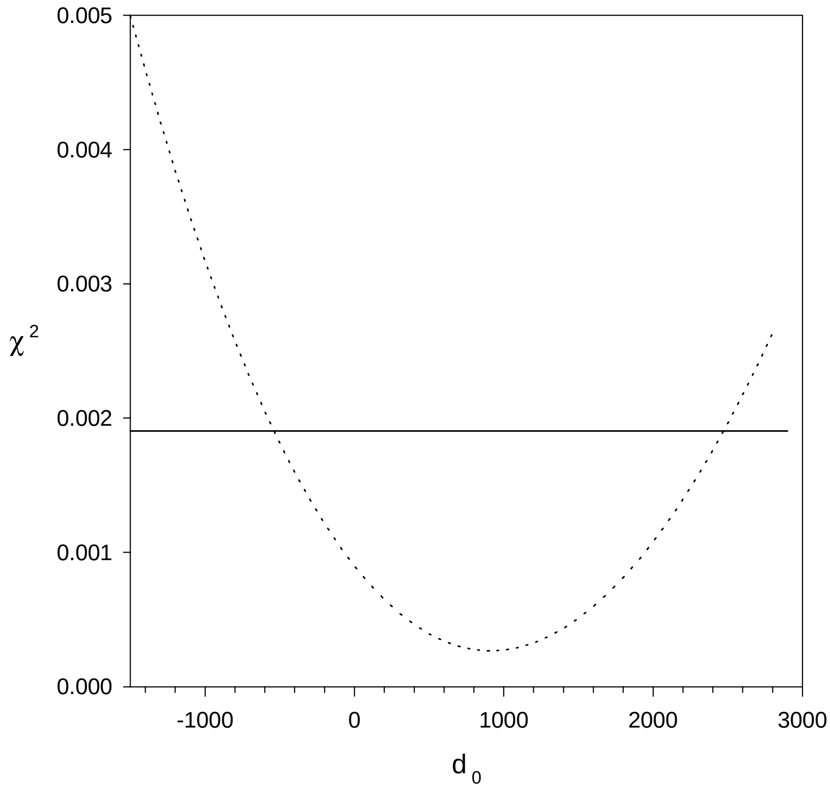

Figure 3: The quantity χ^2 as a function of $d_0^{(4)}$ (dashed curve) for the four-loop contributions to Γ after input of the Padé estimate $c_0^{(4)} = 188$. The straight solid line represents the three-loop χ^2. The criterion that the four-loop χ^2 is smaller three-loop χ^2 constrains $d_0^{(4)}$ to the region between the intersection points of the curves.